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Distributed piezoelectric modal sensors for circular plates

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Abstract

In this note, we deal with finding the shape of distributed piezoelectric modal sensors for circular plates with polar symmetric boundary conditions. The problem is treated by an optimization approach, where a binary function is used to model the design variable: the polarization profile of the piezoelectric layer. The numerical procedure proposed here allows us to find polarization profiles which take on two values only, i.e. either positive or negative polarization, that isolate particular vibration modes in the frequency domain.

1. Introduction

Modal sensors/actuators [1] (hereafter MSA) are those which measure/excite a single mode of a structure, but are not sensitive to the rest of the modes. Several applications on MSA can be found in Ref. [2]. In active control, for instance, the use of MSA reduces the spillover problem by filtering annoying high-frequency modes that affect the stability of closed-loop systems. The well-known reciprocal property of piezoelectric materials remains valid between a modal sensor and a modal actuator [1], that is to say, the sensor shape that observes a particular mode (modal sensor) is the same as the actuator shape that excites that particular mode (modal actuator); hence, in terms of the design we just have to focus on one of them. The sensors and actuators may consist of a number of discrete transducers or a distributed transducer material. In the first case, both the location and the gain have to be determined for each [3,4]. Unlike discrete transducers, distributed transducers are designed by shaping the surface electrode (sometimes also the polarization profile) of the piezoelectric layers, allowing us to determine both location and gain at the same time and in this way reducing the signal processing requirements.

To design distributed modal sensors for plate-type structures, two main variables (among others) must be taken into account: the effective surface electrode that is modeled by a binary function $\chi_e(x,y)$ (χ_e equals 1 if (x,y) is covered by an electrode: otherwise, it is 0), and the polarization profile of the piezoelectric sensor layer, modeled by another binary function $\chi_p(x,y)$ (which typically takes on the values -1 and 1 only).

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The problem of finding distributed MSA for beams is confined to computing the normalized surface electrode width, $\mathscr{F}(x)$ (given by the integral of $\chi_e(x,y)\chi_p(x,y)$ along the y-axis direction), with x being the longitudinal axis of the structure. For such cases, it is proved both theoretically and experimentally in Ref. [1] that either the modal actuator profile or the modal sensor profile is found as a constant times the second derivative of that particular mode shape (or the curvature). An appropriate interpretation of such a function $\mathscr{F}(x)$ gives us all the information we need to construct the aforementioned distributed MSA: on the one hand, its absolute value indicates the gain distribution of the transducer and on the other, it forces the polarization profile (positive or negative) of the piezoelectric layers to vary along the x-axis direction in accordance with its profile.

Basically, the condition which allows us to construct distributed MSA is the orthogonality principle among the vibration mode shapes of a structure [1]. This orthogonality principle can be easily proved for beams whichever the boundary conditions considered, as well as for plates with pinned boundary conditions, because in such cases the modes are given by sinusoid functions, and of course, these verify the criterion. However, a general orthogonality principle does not exist for the vibration mode shapes of plates [5,6], because of the complexity of the boundary conditions, and therefore, we cannot state that

$$\int_S \phi_{rj}\phi_{mn}\,\mathrm{d}x\,\mathrm{d}y = 0 \quad \text{for } r \neq m,\ j \neq n, \tag{1}$$

where ϕ_{rj} is the mode shape of the rj-mode and S is the area covered by the piezoelectric layers, is true in general. Of course, a modal orthogonality principle is satisfied among the modes, but including weight functions (both stiffness and mass matrices in the finite element formulation). That is not the orthogonality principle expressed in Eq. (1), which is exactly what we would need.

Many authors (see Ref. [7] and the references therein) have studied the problem of designing distributed MSA for two-dimensional structures, but to date, a systematic way of doing this has not been found. In the pioneering work [1], a way of creating ideal distributed MSA for a four-sided simply supported rectangular plate is suggested. As discussed above, the particular boundary conditions for this situation make it possible to obtain theoretically a family of distributed MSA, but the problem now is that normalized distributed MSA take on values in the continuous interval $[-1, 1]$. One possible physical interpretation is to assume $\chi_e \equiv 1$ and to require a precisely implemented variation in the polarization profile, χ_p, over x and y, but as pointed out in Ref. [6], this could be really difficult to achieve in practice, from a manufacturing point of view.

In line with the model considered in Ref. [7] and using the philosophy of the topology optimization problems (the reader is referred to Ref. [8] for an excellent overview of the method and different applications), a systematic procedure is proposed in Ref. [9] for designing distributed MSA for rectangular plates through an appropriate optimization approach. This approach uses two binary functions to decide which regions of the piezoelectric layers have to be covered by an electrode and which ones not. In the former case, the functions are used to decide which parts of the transducer material must be polarized in the upward direction, and which ones in the opposite direction.

The aim of this paper is to extend this approach to the study of circular plates with polar symmetric boundary conditions. The layout of the paper is as follows: in the next section an optimization approach is proposed and the physics of the problem is briefly discussed. Later on, a mathematical analysis of the problem is carried out showing that optimal solutions actually correspond to entirely covering the layers by an electrode and polarization profiles taking on two values only. Finally, several numerical examples for two case studies are included to illustrate that the topologies obtained make it possible to isolate particular modes in the frequency domain.

2. Formulation of the optimization problem

We consider a thin circular plate with a piezoelectric sensor layer of negligible stiffness and mass compared with the plate bonded to the top surface as shown in Fig. 1.

Concerning the piezoelectric stress/charge constants [10], we will assume that $e_{31} = e_{32} = e$ (i.e. the piezo's charge per unit area is the same in both radial and circumferential directions) and $e_{36} = 0$ (i.e. the sensor's

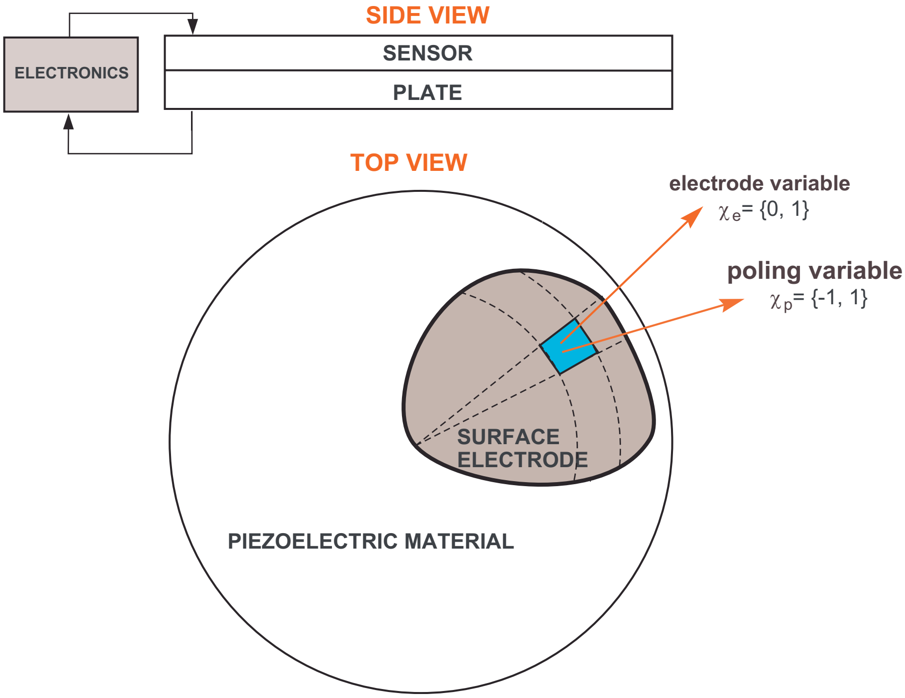

Fig. 1. Design domain.

piezoelectric axes are coincident with the geometric axes of the plate), so that the response of the piezoelectric sensor can be expressed as

$$q(t) = -\frac{(h_p + h_s)}{2} e \int_0^{2\pi} \int_0^R \chi_e \chi_p(r,\theta) \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} \right) r \, \mathrm{d}r \, \mathrm{d}\theta, \tag{2}$$

where q is the sensor output charge, h_p the thickness of the plate, h_s the thickness of the sensor, w the out-of-plane displacement of the plate and R the radius of the plate.

In the case of axisymmetric boundary conditions, the out-of-plane displacement of a circular plate, w, can be written by the modal expansion [5]

$$w(r,\theta,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \phi_{mn}(r) \cos(m\theta) \eta_{mn}(t), \tag{3}$$

where $\phi_{mn}(r,\theta) = a_{mn} J_m(\lambda_{mn} r/R) + b_{mn} I_m(\lambda_{mn} r/R)$ and J_m and I_m are the Bessel function and the modified Bessel function of the first kind, respectively. Inserting Eq. (3) in Eq. (2), we arrive at

$$q(t) = -e \frac{(h_p + h_s)}{2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} B_{mn} \eta_{mn}(t) \tag{4}$$

with

$$B_{mn} = \int_0^{2\pi} \int_0^R \chi_e \chi_p(r,\theta) \left(\phi''_{mn}(\lambda_{mn} r/R) + \frac{1}{r} \phi'_{mn}(\lambda_{mn} r/R) - \frac{m^2}{r^2} \phi_{mn}(\lambda_{mn} r/R) \right) \cos(m\theta) r \, \mathrm{d}r \, \mathrm{d}\theta, \tag{5}$$

such that $\chi_e(r,\theta) \in \{0,1\}$ (to place an electrode or not) and $\chi_p(r,\theta) \in \{-1,1\}$ (positive or negative polarization). Truncating Eq. (4) in the first M modes, the following optimization approach:

$$\text{Maximize}_{\chi_e,\chi_p} \quad B_k(\chi_e, \chi_p) \tag{P}$$

subject to

$$B_j(\chi_e, \chi_p) = 0 \quad \text{for } j = 1, \ldots, M \text{ and } j \neq k \tag{6}$$

is proposed to find an ideal modal sensor that observes the kth mode (i.e. the coefficient B_k is maximized) among the first M modes and filters the rest of them (i.e. the rest of the coefficients B_j with $j \neq k$ are cancelled). For convenience, both the modes and the coefficients are now indexed with the single integer j rather than m and n.

3. Mathematical analysis of the optimization problem (P) and numerical approach

From a mathematical perspective problem (P) is a quite simple optimization problem: maximizing a linear cost subject to linear constraints. However, a difficulty arises due to the fact that the optimization variables χ_e and χ_p take on values on the non-convex sets $\{0, 1\}$ and $\{-1, 1\}$. In order to better explain this point, let us collect both functions χ_e and χ_p together on the single function χ, so that they would then take on values on the (non-convex) set $\{-1, 0, 1\}$. Notice that χ_e and χ_p appear together multiplied both in the cost functional and in the constraints. The optimization problem (P) can be reformulated as

$$\text{Maximize}_{\chi \in \{-1,0,1\}} \quad B_k(\chi) \tag{P}$$

subject to

$$B_j(\chi) = 0 \quad \text{for } j = 1, \ldots, M \text{ and } j \neq k. \tag{7}$$

It is well known that structural optimization problems and material design problems, both formulated as optimization problems in which the optimization variables are functions taking on a finite number of values (and then taking values on a non-convex set), typically lack optimal solutions. It can be observed in those problems that minimizing sequences of designs develop high oscillations between the possible values they may take on, and consequently they do not converge to any admissible design in the proper sense. In practice, optimal solutions show a microgeometry that cannot be described by usual functions taking on a finite number of values. In this situation, a relaxation procedure must be implemented in order to obtain a manageable (relaxed) formulation of the problem, both in a mathematical and a numerical sense. The usual procedure consists of enlarging the set of admissible designs, so as to include all the *mixtures* obtained from the admissible designs for the original problem. A very good account of structural and material optimization from an engineering point of view is [8]. After this discussion, we cannot hope in principle problem (P) to admit optimal solutions, and we should work rather with a relaxed formulation, (RP), of it, namely,

$$\text{Maximize}_{-1 \leqslant \rho \leq 1} \quad B_k(\rho) \tag{RP}$$

subject to

$$B_j(\rho) = 0 \quad \text{for } j = 1, \ldots, M \text{ and } j \neq k. \tag{8}$$

Problem (RP) is actually the same problem as (P), but we have changed the set where we optimize to include any function taking on values between -1 and 1 instead of functions taking on the values $\{-1, 0, 1\}$. Now, based on standard mathematical arguments, we are sure that (RP) is a well-posed problem, i.e. it admits optimal solutions. Further, what is very interesting in this problem is the fact that due to its linear nature, optimal solutions actually take on values -1 or 1 only. This is stated in the following theorem.

Theorem 1. *Any optimal solution for problem* (RP) *takes on the values either* -1 *or* 1.

This theorem has been proved in Ref. [9] in the context of rectangular plates and the reader is referred to Ref. [9] for a more detailed discussion on the mathematical issues of this problem as well as all information concerning the proof.

A direct and very remarkable consequence of Theorem 1 is that we will never find with MSA that there are regions in which we do not place an electrode by using our procedure; we just find a polarization profile (positive or negative) of the piezoelectric MSA distributed throughout the whole domain.

For the numerical simulation we discretize the circular plate in N finite elements (typically $N \gg M$) and obtain the discrete optimization problem:

$$\text{Maximize}_{\boldsymbol{\rho}} : \mathbf{F}_k^{\mathrm{T}} \boldsymbol{\rho} \tag{9}$$

subject to

$$\begin{aligned} &\mathbf{F}_j^{\mathrm{T}} \boldsymbol{\rho} = 0 \quad \text{for } j = 1, \ldots, M \text{ and } j \neq k, \\ &-\mathbf{1} \leqslant \boldsymbol{\rho} \leqslant \mathbf{1}, \end{aligned} \tag{10}$$

where $\{\mathbf{F}_j\}_{j=1,\ldots,M}$ is the family vector of the Laplacian (in polar coordinates) of the first M modes multiplied by r and $\boldsymbol{\rho}$ is the vector of the design variables. This approach has the advantage that both the objective

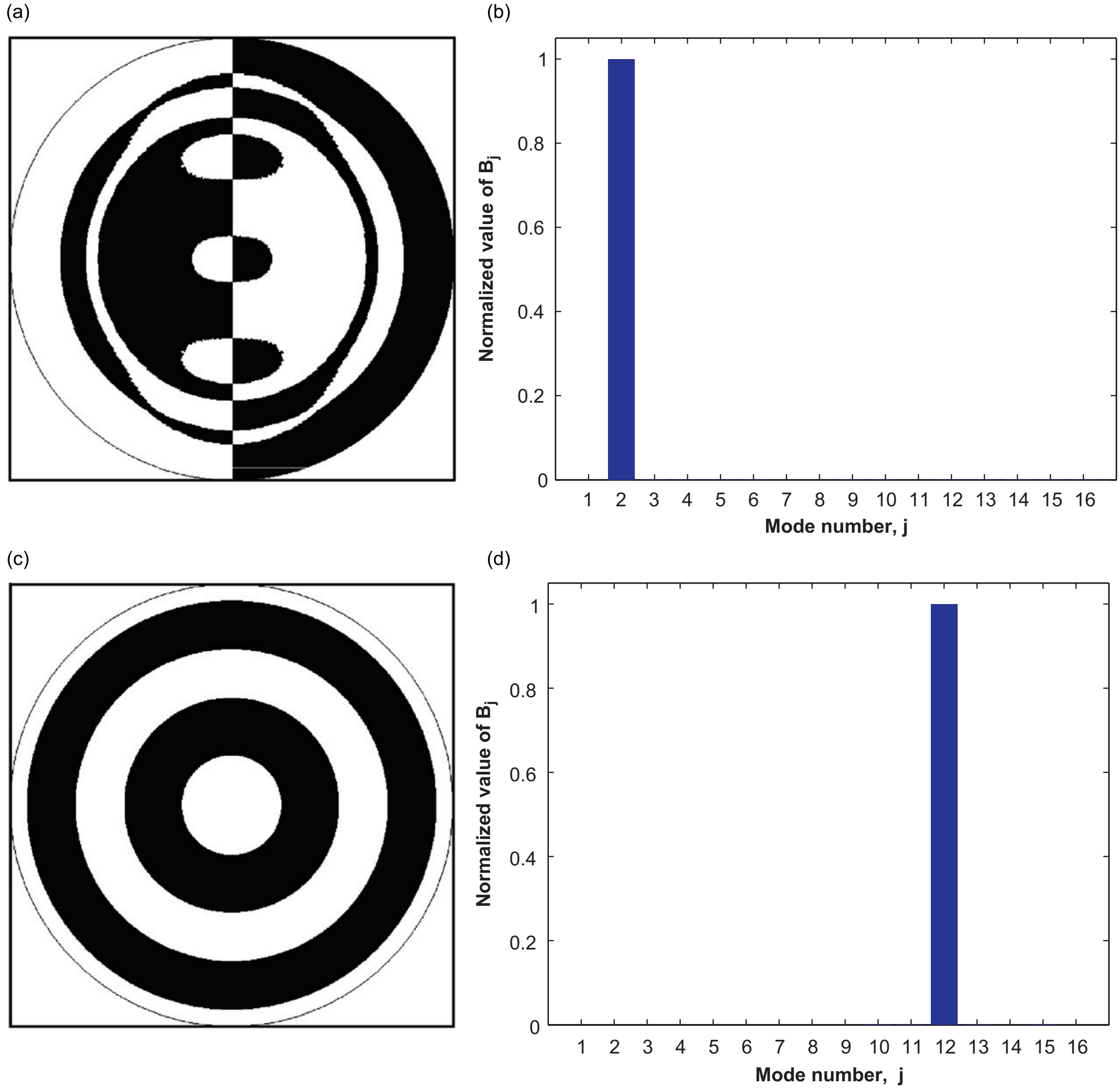

Fig. 2. Polarization profiles that measure the second mode ((a) and (b)) and the twelfth mode ((c) and (d)) for a clamped circular plate when considering $M = 16$.

function and the constraints are linear and hence, it can be easily solved by the simplex method. This very simple optimization procedure makes perfect sense, since we know that optimal solutions for the continuum are extremals of the set of designs verifying the constraints (this is straightforward from Theorem 1), and then, we are actually looking for approximations of these optimal designs on the set of extremals for the discrete problems, by using the simplex method applied to it. This is actually corroborated by the fact that, in our simulations, intermediate values between -1 and 1 in the optimal profiles tend to disappear when we consider

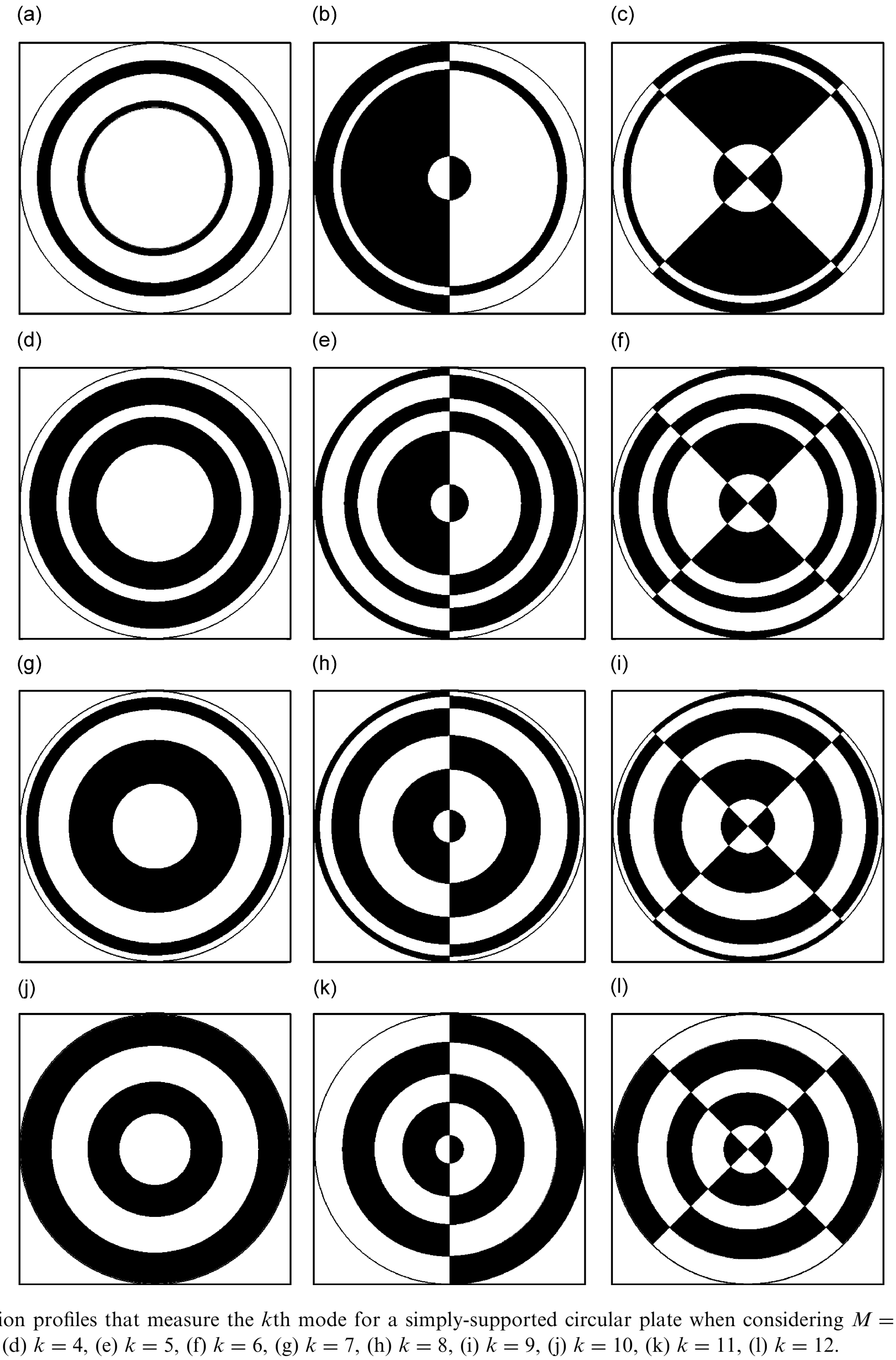

Fig. 3. Polarization profiles that measure the kth mode for a simply-supported circular plate when considering $M = 12$: (a) $k = 1$, (b) $k = 2$, (c) $k = 3$, (d) $k = 4$, (e) $k = 5$, (f) $k = 6$, (g) $k = 7$, (h) $k = 8$, (i) $k = 9$, (j) $k = 10$, (k) $k = 11$, (l) $k = 12$.

finer and finer meshes (see Ref. [9]). Actually for a mesh of 200×200 elements we cannot even notice such intermediate values in any of the examples we have dealt with.

It is worthwhile emphasizing that the applicability of this technique to other more general structures is fairly straightforward. We simply have to find the proper cost functional to the physics of the problem (i.e. the magnitude of the modal response in our approach) and then to compute the mode shapes at interest, either analytically, numerically (by using the subspace iteration method, for instance) or experimentally (using modal data procured by a laser vibrometer, for instance). Finally, the numerical problem is solved by any mathematical programming method as both the objective function and the constraints continue to be linear.

4. Examples

We illustrate our optimization approach through two case studies varying the boundary conditions in a circular plate of unity radius. In all the examples, a mesh of 200×200 elements has been used to compute the mode shapes as well as to run the linear optimization problem.

In this first example, a clamped circular plate is considered. It is well known that, for this particular case, it is possible to express in closed form both natural frequencies and mode shapes [5]. In Fig. 2(a) and (c), the polarization profiles corresponding to a modal sensor sensitive to the second mode and to the 12th mode (see Fig. 2(b) and (d)), respectively, among the first sixteen modes ($M = 16$), are shown. As we can see, at the end of the optimization process, the design variable takes on two values only: $\rho = 1$, represented by a black color, and $\rho = -1$ by a white color, referring to regions with opposite polarization (but, of course, inverting the polarization in all layers, the profile continues to be optimal because it would be the counterpart of the corresponding mode inverted in sign).

To filter any following mode, we merely have to include it by adding a new constraint in the optimization problem, but, as can be expected, the profile changes its topology when the number of modes considered is increased (see Ref. [9] for rectangular plates).

A simply supported circular plate is considered as the second example. As before, the modes can be obtained in closed form (see Ref. [5]). Taking $M = 12$ now, polarization profiles that isolate the first 12 modes for this new situation are shown in Fig. 3.

Depending on the applications, it could be interesting to consider other objective functions or even extra constraints on the curvature, for instance (see Ref. [3] for beam-type structures). This issue will be explored in the near future by the authors.

5. Conclusions

This note has presented a new way to systematically design distributed piezoelectric MSA for circular plates with polar symmetric boundary conditions. A linear optimization approach based on the sensor response is proposed, taking both the effective surface electrode and the polarization profile of the sensor layer as the design variables. It was analytically proved, and numerically corroborated in several examples, that optimized patterns that measure particular vibration modes correspond to the results obtained by entirely covering the layers by an electrode ($\chi_e \equiv 1$) and polarization profiles taking on two values only. We plan to extend this approach to shell-type structures in the near future.

Acknowledgments

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